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# Picture changing operators in supergeometry and superstring theory

Alexander Belopolsky*
University of North Carolina
at Chapel Hill
E-mail: belopols@physics.unc.edu

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**Abstract**

Geometrical meaning of superstring pictures is discussed in details. An off-shell generalization of the picture changing operation and its inverse are constructed. It is demonstrated that the generalised operations are inverse to each other on-shell while off-shell their product is a projection operator.

## 1 Introduction and summary

The notion of pictures was introduced in the early years of superstring theory. Originally pictures appeared as different Fock spaces ($F_1$ and $F_2$) which can be used to represent string states [1, 2]. The spectrum of physical states was shown to be identical in both pictures and the rules for calculating physical amplitudes required to take some states from $F_1$ and others from $F_2$. It was more than a decade later when the physical origin of the pictures was clarified. In their fundamental paper [3] Friedan, Martinec and Shenker (FMS) demonstrated that the choice of a picture is equivalent to a choice of ghost vacuum. They showed that inequivalent representations of bosonic superconformal ghosts can be built using vacuum states with different Bose sea level. Each of these ghost representations would lead to a different BRST complex and therefore to a different description for the physical states. The picture changing operation was introduced as a regular procedure for mapping the physical states (BRST cohomology) from one description to another.

The FMS approach was based on the bosonized description of superconformal ghosts and it provided a very efficient algorithm for calculating the string

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amplitudes. Although the bosonization formulae can greatly simplify the calculations, they tend to obscure the physical meaning of the operators. Thus in addition to bosonic ghost fields $\beta$ and $\gamma$, the FMS formalism introduces auxiliary free fermions $\eta$ and $\xi$ and fermionic "solitons" $e^{\phi}$ and $e^{-\phi}$ whose origin was rather unclear. It was later realized that the solitons can be interpreted as delta functions of the bosonic ghosts $e^{\phi} = \delta(\beta)$ and $e^{-\phi} = \delta(\gamma)$ and they appear in the expressions for the string amplitudes due to the gauge fixing in the functional integral [4, 5]. One of the auxiliary fermions, $\xi$ has also received a simple interpretation as a Heavyside step function of the bosonic antighost field $\xi = \Theta(\beta)$. This elegant interpretation explained the operator product expansions for these fields, but it was still not clear why a delta or step function of a bosonic field produced a fermion [6].

The existence of the inverse picture changing operation was conjectured in the original paper of FMS and the corresponding operator was constructed explicitly by Witten in order to formulate the string field theory of open superstrings [7]. Witten used the operator $Y(z) =: c(z)\partial\xi(z)e^{-2\phi(z)}:$ as an inverse to the FMS picture changing operator $\Gamma(z) = [Q, \xi(z)]$. Although a formal proof was presented in the original paper, the existence of the inverse to the picture changing operation remained controversial since the picture changing operation was known to change the matter contents of a state while $Y(z)$ was expressed in terms of ghost fields exclusively. Recently there was an intriguing attempt to construct a physical state which exists in one picture but not in the other [8, 9]. An existence of such state would definitely be an obstruction to the inverse picture changing operation. A closer look at the proposed state reveals that it is not annihilated by the full superstring BRST operator[1], but it might be possible to modify the theory so that it is still a physical state at least for the zero momentum [10].

The picture changing operation was an essential ingredient of the FMS formalism. At least two different pictures were necessary in order to calculate string amplitudes and to define the SUSY generators. It was long suspected that the need for different picture was an artifact of a particular choice of odd coordinates on the super moduli space [11, 12], but until recently there was no formalism available to express the string amplitudes without first making a choice of odd coordinates. Such formalism was developed by the author in [13, 14] using the geometrical theory of integration on supermanifolds. In the latter paper an analog of the FMS picture changing operation was introduced on the de Rham complex of a supermanifold. Since the FMS picture changing operation proved to be unnecessary in the new formalism, it was not considered in [14] and the relation between the picture changing operations on de Rham and BRST complexes was not properly addressed. In this paper we will attempt to close this gap.

The paper is divided into three parts. The first part is devoted to picture changing operators in supergeometry, the second to picture changing operators in superstring theory and the third provides a bridge between the two applica-

[1] I would like to thank Walter Troost for sharing with me this observation.

tions.

The most important lesson which the supergeometry provides for the string theory is that there are two ghost numbers, an even one and an odd one, in the ghost sector. The ghost number used in the FMS approach is the difference of these two numbers and does not have a separate geometrical meaning. Both fermionic and bosonic ghosts change only the even ghost number and one has to introduce the solitons $\delta(\beta)$ and $\delta(\gamma)$ in order to change the odd ghost number.

The odd ghost number is closely related to the notion of pictures. For each odd ghost number we can construct a separate BRST complex closed under the action of the ghost algebra and the BRST operator. We will show that although these complexes are not isomorphic, an isomorphism exists in cohomology and can be provided by picture changing operators. Picture changing operators and their inverses can also be extended off-shell, but their product will no longer be the identity but a projection operator. It is possible that a projection to the subspace where the picture changing operation is invertible is necessary for a successful construction of a superstring field theory.

# 2 Picture changing operators in supergeometry

In this section we introduce a variant of a super generalization of the de Rham complex of differential forms. For a supermanifold of dimension $n|m$ we will define what will be called singular $r|s$-forms. Possible values for the odd degree of a singular form are $s = 0, \dots, m$ while the odd degree, $r$, can be any integer number if $s$ is not equal to $r$ or $m$; $r \geq 0$ for $s = 0$ and $r \leq n$ for $s = m$. The super de Rham complex, $\Omega^{\cdot|\cdot} = \bigoplus_{r,s} \Omega^{r|s}$ will come equipped with a differential $\mathbf{d}: \ \Omega^{r|s} \to \Omega^{(r+1)|s}$ and the cohomology will be defined for each permissible value of the degree $r$ and $s$. Since the differential does not change the odd degree of a form, the de Rham complex naturally splits into $m+1$ independent subcomplexes each with a different value of $s$. We will refer to these subcomplexes as pictures. Given a non-vanishing odd vector field $\hat{v}$ we will construct an operator $\Gamma_{\hat{v}}$ which will map forms in picture $s$ to forms in picture $s-1$ and commute with the differential. Composition properties of the picture changing operators will be addressed. The existence of the picture changing operators strongly suggests that the cohomology is the same in every picture. We will demonstrate this for a simple case of a $1|1$ dimensional supermanifold for which we will construct an inverse to the picture changing operator on the cohomology.

## 2.1 Singular differential forms on supermanifolds

Differential $r|s$-forms on the supermanifolds were first introduced by Voronov and Zorich in [15–19]. One of the difficulties in their approach was that explicit examples were difficult to obtain. Here we will use their definition with a slight modification to relax the non-singularity condition. By doing that we will be able to produce plenty of forms which were not allowed in the original works.

One of the major benefits will be an explicit description of a large subset of differential forms on supermanifolds closed under the action of the de Rham differential.

A function of $r$ even and $s$ odd tangent vector to a supermanifold,

$$\omega^{(r|s)}(\boldsymbol{v}) = \omega^{(r|s)}(v_1, v_2, \ldots, v_r|\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_s),$$

is called a differential $r|s$ form if it satisfies the following two conditions:

$$\omega^{(r|s)}(J\,\boldsymbol{v}) = \mathrm{Ber}\,J\,\omega^{(r|s)}(\boldsymbol{v}), \quad \text{for any } J \in GL(r|s). \tag{2.1}$$

and

$$\frac{\partial^2 \omega^{(r|s)}(\boldsymbol{v})}{\partial v_F^A \partial v_G^B} + (-1)^{[F][G]+[B]([F]+[G])} \frac{\partial^2 \omega^{(r|s)}(\boldsymbol{v})}{\partial v_G^A \partial v_F^B} = 0, \tag{2.2}$$

where we use the symbol $\boldsymbol{v} = (v_1, v_2, \ldots, v_r|\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_s)$ to refer collectively to the arguments of $\omega^{(r|s)}$. Naturally $v_F^A$ denotes the $A$-th component of the $F$-th vector, so indices $F$ and $G$ run from 1 to $r|s$ and $A$ and $B$—from 1 to the dimension of the manifold. We will denote the space of differential $r|s$-forms on the supermanifold $M$ by $\Omega^{r|s}(M)$.

The first identity, eq. (2.1) is called the invariance condition and due to this condition we can define an integral of $\omega^{(r|s)}$ over a dimension $(r|s)$ submanifold $N \in M$

$$\int\limits_N \omega^{(r|s)} = \int\limits_N \mathcal{D}(t^1, \ldots, t^r|\hat{t}^1, \ldots, \hat{t}^s)\, \omega^{(r|s)} \left( x(t); \frac{\partial x}{\partial t^1}, \ldots, \frac{\partial x}{\partial t^r} \middle| \frac{\partial x}{\partial \hat{t}^1}, \ldots, \frac{\partial x}{\partial \hat{t}^s} \right),$$

which does not depend on the parameterization $\{t^F\}$ of $N$.

The second identity is essential[2] for Stokes theorem. Let now $N$ be an $(r+1)|s$-dimensional submanifold with a boundary $\partial N$, which is an $r|s$-dimensional submanifold. The Stokes theorem states that

$$\int\limits_N \mathbf{d}\omega^{(r|s)} = \int\limits_{\partial N} \omega^{(r|s)},$$

where $\mathbf{d}\omega^{(r|s)}$ is an $(r+1)|s$-form given by

$$[\mathbf{d}\omega^{(r|s)}](x; v_1, v_2, \ldots, v_{r+1}|\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_s) = \\ (-1)^r v_{r+1}^A \frac{\partial \omega^{(r|s)}(x; \boldsymbol{v})}{\partial x^A} - (-1)^{[A][F]} v_F^B \frac{\partial^2 \omega(x; \boldsymbol{v})}{\partial x^B \partial v_F^A}.$$

A canonical example of an $r|s$-form can be found using a set

$$\boldsymbol{v}^\vee = (v_1^\vee, v_2^\vee, \ldots, v_r^\vee|\hat{v}_1^\vee, \hat{v}_2^\vee, \ldots, \hat{v}_s^\vee)$$

[2]Strictly speaking Stokes theorem would be still valid even without eq. (2.2), but in this case $\mathbf{d}\omega$ is no longer a differential form and depends on second derivatives $\ddot{x}(t)$. See [20] for details.

of $r$ even and $s$ odd covectors:

$$\omega(\boldsymbol{v}) = \mathrm{Ber}\|v_G^\vee(v_F)\|. \tag{2.3}$$

In the original formulation the $r|s$-forms were required to be nonsingular unless $\mathrm{rank}(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_s) < s$. Obviously this is not true for the example given in eq. (2.3), where the form has a pole whenever one of the odd argument $\hat{v}$ belongs to the null space of of the defining covectors, or $\hat{v}_1^\vee(\hat{v}) = \hat{v}_2^\vee(\hat{v}) = \cdots = \hat{v}_s^\vee(\hat{v}) = 0$.

## 2.2 Singular $r|s$ forms

Equations (2.1) and (2.2) have to be satisfied at every point of the supermanifold and they can be solved independently for different points. Let us forget for a moment about the dependence of $\omega$ on the point where it is evaluated and concentrate on the solutions to (2.1) and (2.2) at a point. As a starting point we will take the solution given by eq. (2.3) and produce other solutions by applying certain operations to it.

Our first operation will map an $r|s$-form to an $(r+1)|s$-form by taking an exterior product with a covector $v^\vee$:

$$[\mathbf{e}_{v^\vee}\,\omega](v_1, v_2, \ldots, v_{r+1}|\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_r) = \\ (-1)^r \left[ v^\vee(v_{r+1}) - (-1)^{[v^\vee][F]} v^\vee(v_F) v_{r+1}^A \frac{\partial}{\partial v_F^A} \right] \omega(\boldsymbol{v}).$$

One can check that if $\omega$ satisfies eqs. (2.1) and (2.2), then so does $\mathbf{e}_{v^\vee}\,\omega$. Exterior product operators commute (in the super sense) if their parity is defined as opposite to the parity of the corresponding covector

$$[\mathbf{e}_{v^\vee}] = [v^\vee] + \bar{1}.$$

Using even covectors[3] we will not produce new forms by applying $\mathbf{e}_{v^\vee}$ to the canonical form in (2.1) since

$$\mathrm{Ber}\|v_G^\vee(v_F)\| = \mathbf{e}_{v_1^\vee}\mathbf{e}_{v_2^\vee}\cdots\mathbf{e}_{v_r^\vee}(\det\|\hat{v}_i^\vee(\hat{v}_j\|)^{-1}, \tag{2.4}$$

and by applying extra $\mathbf{e}_{v^\vee}$ we will obtain forms of the same type but with higher even degrees. New forms appear when odd covectors are used to take an exterior product. The most interesting feature of these new forms is that their even dimension is not bounded by the even dimension of the manifold. Indeed, operators $\mathbf{e}_{\hat{v}_1^\vee}$ and $\mathbf{e}_{\hat{v}_2^\vee}$ commute rather than anticommute and therefore can be iterated indefinitely. It will be convenient to introduce a shorter notation for the exterior product

$$\mathbf{e}_{v^\vee}\omega = \tilde{v}^\vee\,\omega, \tag{2.5}$$

[3]We will use different conventions for the parity of covectors and 1-forms. Even covector is the one which takes even values on even and odd values on odd vectors while an odd covector does otherwise. The parity of a 1-form is opposite to the parity of the corresponding covector.

where $\tilde{v}^\vee$ is the 1-form corresponding to the covector $v^\vee$. Note that we have to make a distinction between covectors and 1-forms since the parity of a one form is opposite to the parity of the corresponding covector.

The second operation that we will define will decrease the degree of the form by substituting a given vector for one of its arguments. In doing so we have two inequivalent choices—to substitute the given vector for an even or for an odd argument. Substituting a vector for an odd argument will change the singularity of the form and operations that change the singularity will be considered in the next section. We define the inner product with a vector $v$ as follows

$$[\mathbf{i}_v\omega](v_1, v_2, \ldots, v_{r-1}|\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_s) = \omega(v, v_1, v_2, \ldots, v_{r-1}|\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_s).$$

We can use odd vectors as well as even in the inner product. Similarly to the exterior product, inner product operators commute (in the super sense) if their parity is defined as opposite to the parity of the corresponding vector

$$[\mathbf{i}_v] = [v] + \bar{1}.$$

Note that inner products with odd vectors can be iterated and $(\mathbf{i}_{\hat{v}})^k$ is non-zero for any $k$. Of course, the iterated operator is only defined on forms with even degree $r \geq k$, but it will be useful to extend the space of singular forms by forms with negative even degree so that $(\mathbf{i}_{\hat{v}})^k$ is defined on every form for arbitrarily large $k$. We will show that all the form operations like exterior product, differential and Lie derivative can be defined on the extended space.

Operators of exterior and interior products form together the super Clifford algebra

$$[\mathbf{e}_{v_1^\vee}, \mathbf{e}_{v_2^\vee}] = [\mathbf{i}_{v_1}, \mathbf{i}_{v_2}] = 0, \quad [\mathbf{i}_v, \mathbf{e}_{v^\vee}] = v^\vee(v). \tag{2.6}$$

We can describe the space of singular forms with a given singularity as a free module over the super Clifford algebra (2.6) generated from the vacuum state $\omega(\boldsymbol{v}) = (\det\|\hat{v}_i^\vee(\hat{v}_j\|)^{-1}$. For even $v$ and $v^\vee$, $\mathbf{i}_v$ is an annihilation operator,

$$\mathbf{i}_v\,(\det\|\hat{v}_i^\vee(\hat{v}_j\|)^{-1} = 0,$$

and $\mathbf{e}_{v^\vee}$ is a creation operator. For odd $\hat{v}$ and $\hat{v}^\vee$, the situation is a bit more complicated. The vacuum is annihilated by $\mathbf{e}_{\hat{v}^\vee}$ if $\hat{v}^\vee$ belongs to the space spanned by $\hat{v}_i^\vee$ and by $\mathbf{i}_{\hat{v}}$ if all $\hat{v}_i^\vee(\hat{v}) = 0$:

$$\mathbf{e}_{\hat{v}^\vee}\,(\det\|\hat{v}_i^\vee(\hat{v}_j)\|)^{-1} = 0 \qquad \text{if } \hat{v}^\vee = \textstyle\sum_i a_i\,\hat{v}_i^\vee, \tag{2.7}$$

$$\mathbf{i}_{\hat{v}}\,(\det\|\hat{v}_i^\vee(\hat{v}_j)\|)^{-1} = 0 \qquad \text{if } \hat{v}_i^\vee(\hat{v}) = 0 \text{ for all } i = 1, \ldots, s. \tag{2.8}$$

In the two limiting cases, when the odd degree $s$ of the vacuum state is either zero or equals the odd dimension $m$ of the supermanifold, the even degree, $r$, of the forms is bounded from one side. In the first case the vacuum is annihilated by all $\mathbf{i}_{\hat{v}}$ and therefore $r \geq 0$; in the second case, the vacuum is annihilated by all $\mathbf{e}_{\hat{v}^\vee}$ and the odd degree of the forms is bounded by the even degree of the supermanifold, $r \leq n$. For all the intermediate values of $s$, $r$ can be any integer number.

## 2.3 Creating the singularity

In the previous section we have shown how to generate a whole space of differential forms by applying some creation operators to a given form. In this construction all the resulting forms had the same odd dimension and the same singularity as the original one. Now we want to introduce operators that will change them both.

We define

$$[\delta(\mathbf{i}_{\hat v})\omega](v_1, v_2, \ldots, v_r|\hat v_1, \hat v_2, \ldots, \hat v_{s-1}) = \\ (-1)^r \omega(v_1, v_2, \ldots, v_r|\hat v, \hat v_1, \hat v_2, \ldots, \hat v_{s-1}),$$

and

$$[\delta(\mathbf{e}_{\hat v^\vee})\omega](v_1, v_2, \ldots, v_r|\hat v_1, \hat v_2, \ldots, \hat v_{s+1}) = \\ \frac{1}{\hat v^\vee(\hat v_{s+1})}\,\omega\left(\cdots, v_F - \frac{\hat v^\vee(v_F)\, v_{s+1}}{\hat v^\vee(\hat v_{s+1})}, \cdots\right),$$

where $F = 1, \ldots, r|s$.

The following identities, which easily follow from definitions partially justify the use of the delta function notation

$$\delta(\mathbf{i}_{\hat v})\,\mathbf{i}_{\hat v} = \mathbf{i}_{\hat v}\,\delta(\mathbf{i}_{\hat v}) = 0, \qquad \delta(\mathbf{e}_{\hat v^\vee})\,\mathbf{e}_{\hat v^\vee} = \mathbf{e}_{\hat v^\vee}\,\delta(\mathbf{e}_{\hat v^\vee}) = 0.$$

It will be useful to define also $\delta(\mathbf{i}_v)$ and $\delta(\mathbf{e}_{v^\vee})$ for even $v$ and $v^\vee$. In this case $\mathbf{i}_v$ and $\mathbf{e}_{v^\vee}$ are odd operators and it is natural to assume that $\delta(\mathbf{i}_v) = \mathbf{i}_v$ and $\delta(\mathbf{e}_{v^\vee}) = \mathbf{e}_{v^\vee}$. Let us now list some of the commutation properties of the new operators

$$\begin{aligned} \delta(\mathbf{i}_u)\,\delta(\mathbf{i}_v) &= -\delta(\mathbf{i}_v)\,\delta(\mathbf{i}_u) & \mathbf{i}_u\,\delta(\mathbf{i}_v) &= -(-1)^{[u]}\delta(\mathbf{i}_v)\,\mathbf{i}_u \\ \delta(\mathbf{e}_{u^\vee})\,\delta(\mathbf{e}_{v^\vee}) &= -\delta(\mathbf{e}_{v^\vee})\,\delta(\mathbf{e}_{u^\vee}) & \mathbf{e}_{u^\vee}\,\delta(\mathbf{e}_{v^\vee}) &= -(-1)^{[u^\vee]}\delta(\mathbf{e}_{v^\vee})\,\mathbf{e}_{u^\vee}. \end{aligned}$$

This shows that all these operators will become supercommutative if we assign odd parity to $\delta(\mathbf{i}_v)$ and $\mathbf{e}_{v^\vee}$ no matter what parity $v$ and $v^\vee$ have.

$$[\delta(\mathbf{i}_v)] = [\delta(\mathbf{e}_{v^\vee})] = \bar{1}.$$

Using $\delta(\mathbf{e}_{\hat v^\vee})$ one can generate all odd vacuum forms from $\omega^{(0|0)} = 1$. Indeed, one can show that

$$(\det\|\hat v_i^\vee(\hat v_j)\|)^{-1} = \delta(\mathbf{e}_{\hat v_1^\vee})\,\delta(\mathbf{e}_{\hat v_2^\vee})\,\cdots\,\delta(\mathbf{e}_{\hat v_s^\vee})\,1,$$

or extending the convention of eq. (2.5), we can write

$$(\det\|\hat v_i^\vee(\hat v_j)\|)^{-1} = \delta(\tilde{\hat v}_1^\vee)\,\delta(\tilde{\hat v}_2^\vee)\cdots\delta(\tilde{\hat v}_s^\vee).$$

Furthermore, using the convention that the delta function is identity on the odd arguments, we can rewrite the form in eq. (2.4) as

$$\omega(\boldsymbol{v}) = \mathrm{Ber}\|v_G^\vee(v_F)\| = \prod_{G=1}^{r|s} \delta(\tilde v_G^\vee).$$

## 2.4 Commutation relations

As it was stated above, operators $\mathbf{e}_{v^\vee}$ and their delta functions super commute and so do $\mathbf{i}_v$ and $\delta(\mathbf{i}_v)$. Now we will analyze their mixed products and the products involving two delta functions.

As it is the case with ordinary delta functions the product of two delta symbols with the same argument leads to divergences. On the other hand if arguments are linearly independent then the compositions of the corresponding operators are finite and satisfy the following identities

$$\delta(\mathbf{i}_{\hat{v}_1+\hat{v}_2})\,\delta(\mathbf{i}_{\hat{v}_2}) = \delta(\mathbf{i}_{\hat{v}_1})\,\delta(\mathbf{i}_{\hat{v}_2}) \qquad \delta(\mathbf{e}_{\hat{v}_1^\vee+\hat{v}_2^\vee})\,\delta(\mathbf{e}_{\hat{v}_2^\vee}) = \delta(\mathbf{e}_{\hat{v}_1^\vee})\,\delta(\mathbf{e}_{\hat{v}_2^\vee}). \tag{2.9}$$

Operators $\delta(\mathbf{e}_{\hat{v}^\vee})$ and $\delta(\mathbf{i}_{\hat{v}})$ do not satisfy the same commutation relations as $\mathbf{e}_{\hat{v}^\vee}$ and $\mathbf{i}_{\hat{v}}$, in fact their commutators cannot even be defined except for the trivial case $\hat{v}^\vee(\hat{v}) = 0$ when they commute. However, the following weaker form of the relation $[\mathbf{e}_{v^\vee}, \mathbf{i}_{\hat{v}}] = \hat{v}^\vee(\hat{v})$, which states that

$$\mathbf{e}_{\hat{v}^\vee}\,\mathbf{i}_{\hat{v}}\,\omega = \hat{v}^\vee(\hat{v})\,\omega \qquad\qquad \text{if } \mathbf{e}_{\hat{v}^\vee}\,\omega = 0$$

and

$$\mathbf{i}_{\hat{v}}\,\mathbf{e}_{v^\vee}\,\omega = \hat{v}^\vee(\hat{v})\,\omega \qquad\qquad \text{if } \mathbf{i}_{\hat{v}}\,\omega = 0$$

has an analog for $\delta(\mathbf{e}_{\hat{v}^\vee})$ and $\delta(\mathbf{i}_{\hat{v}})$:

$$\delta(\mathbf{e}_{\hat{v}^\vee})\,\delta(\mathbf{i}_{\hat{v}})\,\omega = \frac{1}{\hat{v}^\vee(\hat{v})}\,\omega \qquad\qquad \text{if } \mathbf{e}_{\hat{v}^\vee}\,\omega = 0 \tag{2.10}$$

and

$$\delta(\mathbf{i}_{\hat{v}})\,\delta(\mathbf{e}_{v^\vee})\,\omega = \frac{1}{\hat{v}^\vee(\hat{v})}\,\omega \qquad\qquad \text{if } \mathbf{i}_{\hat{v}}\,\omega = 0. \tag{2.11}$$

The appearance of $\hat{v}^\vee(\hat{v})$ in the denominator is consistent with the fact that delta function is homogeneous of degree $-1$.

It follows easily from the definitions that the (anti)commutators of $\mathbf{e}_{\hat{v}^\vee}$ with $\delta(\mathbf{i}_{\hat{v}})$ and $\mathbf{i}_{\hat{v}}$ with $\delta(\mathbf{e}_{\hat{v}^\vee})$ are both proportional to $\hat{v}^\vee(\hat{v})$ with the operator factor depending only on the argument of the delta function. This observation allows us to define new operators which we will denote by $\delta'(\mathbf{i}_{\hat{v}})$ and $\delta'(\mathbf{e}_{\hat{v}^\vee})$

$$[\mathbf{e}_{\hat{v}^\vee}, \delta(\mathbf{i}_{\hat{v}})] = \hat{v}^\vee(\hat{v})\,\delta'(\mathbf{i}_{\hat{v}}) \qquad [\mathbf{i}_{\hat{v}}, \delta(\mathbf{e}_{\hat{v}^\vee})] = \hat{v}^\vee(\hat{v})\,\delta'(\mathbf{e}_{\hat{v}^\vee}), \tag{2.12}$$

repeating the logic we can define operators with multiple derivatives of delta functions

$$[\mathbf{e}_{\hat{v}^\vee}, \delta^{(n)}(\mathbf{i}_{\hat{v}})] = \hat{v}^\vee(v)\,\delta^{(n+1)}(\mathbf{i}_{\hat{v}}) \qquad [\mathbf{i}_{\hat{v}}, \delta^{(n)}(\mathbf{e}_{\hat{v}^\vee})] = \hat{v}^\vee(v)\,\delta^{(n+1)}(\mathbf{e}_{\hat{v}^\vee}).$$

The degrees of the resulting operators are given by

$$\deg(\delta^{(n)}(\mathbf{i}_{\hat{v}})) = n|(-1) \qquad\qquad \deg(\delta^{(n)}(\mathbf{e}_{\hat{v}^\vee})) = (-n)|1.$$

Recall that for even $v$ and $v^\vee$, and therefore odd $\mathbf{i}_v$ and $\mathbf{e}_{v^\vee}$ the delta functions of the operators were identified with the operators themselves. Identities in

(2.12) would still hold under this convention since $\delta'(\mathbf{i}_v) = \delta'(\mathbf{e}_{v^\vee}) = 1$ and both identities would become equivalent to the last identity in (2.6).

We will also use derivatives of delta functions to write down the singular forms without explicit use of $\mathbf{i}_{\hat{v}}$ operators. Recall that the space of singular forms was defined as a module generated by all possible $\mathbf{e}_{v^\vee}$ and $\mathbf{i}_v$ operators from a vacuum $0|s$-form which can be written as $\delta(\tilde{\hat{v}}_1^\vee)\cdots\delta(\tilde{\hat{v}}_s^\vee)$. Each application of $\mathbf{i}_{\hat{v}}$ results in a derivative of a delta function

$$\mathbf{i}_{\hat{v}}\delta(\tilde{\hat{v}}_1^\vee)\cdots\delta(\tilde{\hat{v}}_s^\vee) = \sum_{k=1}^{s} \hat{v}^\vee(\hat{v})\,\delta(\tilde{\hat{v}}_1^\vee)\cdots\delta'(\tilde{\hat{v}}_k^\vee)\cdots\delta(\tilde{\hat{v}}_s^\vee).$$

One of the benefits of this new notation is that eq. (2.8) becomes a tautology.

## 2.5 The differential and the Lie derivative

We have briefly introduced the de Rham differential in section 2.2. Let us repeat the definition. The de Rham differential, $\mathbf{d}$, is a nilpotent differential operator which maps $\Omega^{r|s}(M)$ to $\Omega^{(r+1)|s}(M)$ and can be written in coordinates as

$$[\mathbf{d}\omega](x; v_1, v_2, \ldots, v_{r+1}|\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_s) = \\ (-1)^r v_{r+1}^A \frac{\partial\omega(x;\boldsymbol{v})}{\partial x^A} - (-1)^{[A][F]} v_F^B \frac{\partial^2\omega(x;\boldsymbol{v})}{\partial x^B \partial v_F^A}.$$

Another important differential operation on the differential forms is the Lie derivative along a vector field. The Lie derivative can be defined as usual by differentiating the drag of a form produced by the vector field and this would produce the following explicit formula:

$$\mathcal{L}_v\omega(x;\boldsymbol{v}) = v^A \frac{\partial\omega(x;\boldsymbol{v})}{\partial x^A} + (-1)^{[F][v]}\, v_F^B\, \frac{\partial v^A}{\partial x^B} \frac{\partial\omega(x;\boldsymbol{v})}{\partial v_F^A}.$$

In a complete analogy with the purely even case, the differential and the Lie derivative satisfy the following super commutator relations with the inner product:

$$[\mathcal{L}_v, \mathbf{i}_u] = \mathbf{i}_{[v,u]}, \tag{2.13}$$

and

$$\mathcal{L}_v = [\mathbf{d}, \mathbf{i}_v]. \tag{2.14}$$

These identities make it possible to extend the definition of $\mathbf{d}$ and $\mathcal{L}_v$ to the differential forms of negative degrees which can be generated by repeated application of $\mathbf{i}_{\hat{v}}$ with odd $\hat{v}$.

For our applications we will also need the commutation relations of $\mathbf{d}$ and $\mathcal{L}_v$ with the derivatives of delta function of the inner product operator

$$[\mathcal{L}_{\hat{v}}, \delta^{(n)}(\mathbf{i}_{\hat{v}})] = \delta^{(n+1)}(\mathbf{i}_{\hat{v}})\,\mathbf{i}_{[\hat{v},\hat{v}]}, \tag{2.15}$$

$$[\mathbf{d}, \delta^{(n)}(\mathbf{i}_{\hat{v}})] = \delta^{(n+1)}(\mathbf{i}_{\hat{v}})\,\mathcal{L}_{\hat{v}} + \frac{1}{2}\delta^{(n+2)}(\mathbf{i}_{\hat{v}})\,\mathbf{i}_{[\hat{v},\hat{v}]}, \tag{2.16}$$

where $[\hat{v}, \hat{v}] = 2\,\hat{v}^2$ is the super Lie bracket (anticommutator) of the odd vector field $\hat{v}$ with itself. Note that the all the commutation relations involving delta functions would have exactly the same form if the delta function was replaced with an analytical function.

## 2.6 Picture changing operators

Let $\hat{v}$ be a non-vanishing odd vector field on the supermanifold. We define the picture changing operator associated with $\hat{v}$ as follows

$$\Gamma_{\hat{v}} = \frac{1}{2}\big(\delta(\mathbf{i}_{\hat{v}})\,\mathcal{L}_{\hat{v}} - \mathcal{L}_{\hat{v}}\,\delta(\mathbf{i}_{\hat{v}})\big) = \delta(\mathbf{i}_{\hat{v}})\,\mathcal{L}_{\hat{v}} + \delta'(\mathbf{i}_{\hat{v}})\,\mathbf{i}_{\hat{v}^2}. \tag{2.17}$$

The picture changing operator commutes with the de Rham differential

$$[\mathbf{d}, \Gamma_{\hat{v}}] = 0.$$

This property of $\Gamma_{\hat{v}}$ easily follows from the commutation relations (2.15), (2.16) and (2.13), but it is useful to give the following empirical reason. The picture changing operator can be formally written as

$$\Gamma_{\hat{v}} = [\mathbf{d}, \Theta(\mathbf{i}_{\hat{v}})], \tag{2.18}$$

where $\Theta(x)$ is the Heavyside step function $\Theta'(x) = \delta(x)$. Since $\mathbf{d}^2 = 0$ the vanishing of the commutator $[\mathbf{d}, \Gamma_{\hat{v}}]$ trivially follows from eq. (2.18). Note that although eq. (2.18) expresses $\Gamma_{\hat{v}}$ as a $\mathbf{d}$-exact operator, this does not mean that it would vanish in cohomology because $\Theta(\mathbf{i}_{\hat{v}})$ cannot be defined as an operator within the de Rham complex.

Since picture changing operators contain $\delta(\mathbf{i}_{\hat{v}})$, they cannot be composed unless their arguments are linearly independent at every point of the supermanifold. For two such linearly independent vector fields, $\hat{v}_1$ and $\hat{v}_2$, it is interesting two find the commutator of their corresponding picture changing operators. It turns out that the commutator can be written as

$$[\Gamma_{\hat{v}_1}, \Gamma_{\hat{v}_2}] = [\mathbf{d}, \delta(\mathbf{i}_{\hat{v}_1})\,\delta(\mathbf{i}_{\hat{v}_2})\mathbf{i}_{[\hat{v}_1, \hat{v}_2]}], \tag{2.19}$$

which means that on the cohomology, picture changing operators form a commutative algebra.

It is natural to define the higher order picture changing operators as symmetrized products

$$\Gamma^{(n)}_{\hat{v}_1\cdots\hat{v}_n} = \sum_{\sigma\in\mathfrak{S}_n} \Gamma_{\hat{v}_{\sigma(1)}}\cdots\Gamma_{\hat{v}_{\sigma(n)}}.$$

It follows from eq. (2.19) that

$$\Gamma^{(n+1)}_{\hat{v}_1\cdots\hat{v}_{n+1}} = \Gamma_{\hat{v}_{n+1}}\,\Gamma^{(n)}_{\hat{v}_1\cdots\hat{v}_n} + [\mathbf{d}, \cdots]. \tag{2.20}$$

## 2.7 A toy problem: $\Omega^{\cdot|\cdot}(S^{1|1})$

In order to illustrate the properties of the picture changing operators, let us analyze the de Rham complex of the simplest compact supermanifold—the supercircle. In fact there are two nonequivalent compact supermanifolds of dimension $1|1$, one with a periodic odd coordinate and the other with an antiperiodic one, but the analysis will be exactly the same for both cases. Let $x$ and $\xi$ be the coordinates on $S^{1|1}$ such that $(x+2\,\pi,\xi)=(x,\pm\xi)$. Since the odd dimension of $S^{1|1}$ is one, we have only two pictures: $s=0$ and $s=1$. The de Rham complex in picture $s=0$ is infinite to the right ($r\geq 0$) and in picture $s=1$ it is infinite to the left ($r\leq 1$). This situation is depicted in the following diagram

$$\begin{tikzcd}
 & \boxed{0} & \boxed{1} & \boxed{2} & \boxed{2} & \cdots \\
 & 0 \arrow[r, "\mathbf{d}"] & \Omega^{0|0} \arrow[r, "\mathbf{d}"] & \Omega^{1|0} \arrow[r, "\mathbf{d}"] & \Omega^{2|0} \arrow[r, "\mathbf{d}"] & \cdots \\
\cdots \arrow[r, "\mathbf{d}"] & \Omega^{(-1)|1} \arrow[r, "\mathbf{d}"] & \Omega^{0|1} \arrow[r, "\mathbf{d}"] \arrow[u, "\Gamma"'] & \Omega^{1|1} \arrow[r, "\mathbf{d}"] \arrow[u, "\Gamma"'] & 0 & \\
\cdots & \boxed{2} & \boxed{2} & \boxed{1} & \boxed{0} &
\end{tikzcd} \tag{2.21}$$

Differential forms in picture $s=0$ can be written as polynomials in $\mathrm{d}x$ and $\mathrm{d}\xi$, so the space $\Omega^{r|0}$ is spanned by

$$\omega_1^{(r|0)}=\mathrm{d}x(\mathrm{d}\xi)^{r-1}f_1(x,\xi)$$

and

$$\omega_2^{(r|0)}=(\mathrm{d}\xi)^{r}f_2(x,\xi).$$

Similarly $\Omega^{r|1}$ is spanned by

$$\omega_1^{(r|1)}=\mathrm{d}x\delta^{(1-r)}(\mathrm{d}\xi)g_1(x,\xi)$$

and

$$\omega_2^{(r|1)}=\delta^{(-r)}(\mathrm{d}\xi)g_2(x,\xi).$$

This shows that the dimensions of $\Omega^{r|s}(S^{1|1})$ at a point are as given by boxed numbers in the diagram (2.21). One can easily calculate the differential of these forms and find that the only nonzero cohomology classes can be represented by 1, $\mathrm{d}x$, $\xi\,\delta(\mathrm{d}\xi)$ and $\mathrm{d}x\,\delta(\mathrm{d}\xi)$.

$$H^{r|s}(S^{1|1})=\begin{cases}\mathbb{R} & \text{for } r=0,1 \text{ and } s=0,1\\ 0 & \text{otherwise}\end{cases}$$

Note that it is only due to the presence of negative degree forms that the closed $0|1$ form $\delta(\mathrm{d}\xi)$ is trivial and the cohomology is the same in the two pictures:

$$\delta(\mathrm{d}\xi)=-\mathbf{d}(\xi\,\delta'(\mathrm{d}\xi)).$$

It follows easily from the dimensional analysis that picture changing operator is not invertible. Nevertheless it is possible to define an operator $Y$ which is an inverse of $\Gamma$ restricted to a subspace of $\Omega^{\cdot|\cdot}(S^{1|1})$. Moreover, the pair $\Gamma$ and $Y$ induces an isomorphism of the *cohomology* in the two pictures.

According to Shander's theorem [21] any odd vector field is locally equivalent to one of the following two up to a change of coordinates

$$\partial_\xi = \frac{\partial}{\partial \xi},$$

and

$$D = \frac{\partial}{\partial \xi} + \xi\,\frac{\partial}{\partial x}.$$

The second vector field is more interesting in a way that its square is not zero

$$D^2 = \frac{\partial}{\partial x}.$$

Let us start with $\Gamma_{\partial_\xi}$. A simple calculation shows that

$$\Gamma_{\partial_\xi}\xi\,\delta(\mathrm{d}\xi)\,f(x,\xi) = f(x,\xi) \quad \text{and} \quad \Gamma_{\partial_\xi}\mathrm{d}x\,\delta(\mathrm{d}\xi)\,f(x,\xi) = \mathrm{d}x\,\partial_\xi f(x,\xi).$$

The inverse picture changing operator is therefore

$$Y_{\partial_\xi} = \xi\,\delta(\mathbf{e}_{\mathrm{d}\xi}).$$

Indeed,

$$Y_{\partial_\xi} f(x,\xi) = \xi\,\delta(\mathrm{d}\xi)\,f(x,\xi) \quad \text{and} \quad Y_{\partial_\xi}\mathrm{d}x\,\partial_\xi f(x,\xi) = \mathrm{d}x\,\delta(\mathrm{d}\xi)\,f(x,\xi).$$

Similar analysis applies to $\Gamma_D$ only with slightly more complicated formulae. The inverse picture changing operator in this case is given by

$$Y_D = \xi\,\delta(\mathbf{e}_{\mathrm{d}\xi}) + \mathbf{e}_{\mathrm{d}z}\delta'(\mathbf{e}_{\mathrm{d}\xi}), \tag{2.22}$$

which follows from the identities

$$\begin{aligned} Y_D\,f(x,\xi) &= (\xi\,\delta(\mathrm{d}\xi) + \mathrm{d}z\delta'(\mathrm{d}\xi)))\,f(x,\xi),\\ \Gamma_D\,(\xi\,\delta(\mathrm{d}\xi) + \mathrm{d}z\delta'(\mathrm{d}\xi)))\,f(x,\xi) &= f(x,\xi),\\ Y_D\,(\mathrm{d}\xi\,f(x,\xi) + (\mathrm{d}x - \xi\,\mathrm{d}\xi)\,Df(x,\xi)) &= \mathrm{d}x\delta(\mathrm{d}\xi)\,f(x,\xi),\\ \Gamma_D\,\mathrm{d}x\delta(\mathrm{d}\xi)\,f(x,\xi) &= \mathrm{d}\xi\,f(x,\xi) + (\mathrm{d}x - \xi\,\mathrm{d}\xi)\,Df(x,\xi). \end{aligned}$$

The product of $\Gamma$ with $Y$ as operators in $\Omega^{\cdot|\cdot}$ is not the unit operator but a projector. However, these operators induce the following maps in cohomology

$$\begin{aligned} \bar\Gamma &: H^{\cdot|1} \to H^{\cdot|0},\\ \bar Y &: H^{\cdot|0} \to H^{\cdot|1}, \end{aligned}$$

were the second map is the true inverse of the first

$$\bar\Gamma \circ \bar Y = \mathbb{1} \quad \text{on } H^{\cdot|0}$$

and

$$\bar Y \circ \bar\Gamma = \mathbb{1} \quad \text{on } H^{\cdot|1}$$

# 3 Superstring picture changing operators

In this section we will describe the picture changing operators as they appear in string theory. The main idea is to show that the supersting picture changing operators are closely related to the similar operators on the super de Rham complex. Picture changing operators in supersrings were extensively studied by different methods, and even though their geometrical significance was not fully realized, most of the results that we will present here can be found in the literature [5–7, 22–27].

## 3.1 A review of the FMS approach

### 3.1.1 Superconformal ghosts

Superconformal gauge fixing in the string functional integral produces two pairs of ghost fields: fermionic $b$ and $c$ of conformal dimensions $\Delta_b = 2$ and $\Delta_c = -1$ and bosonic $\beta$ and $\gamma$ of conformal dimensions $\Delta_\beta = 3/2$ and $\Delta_\gamma = -1/2$. Their operator products are given by

$$c(z)b(w) = \frac{1}{z-w} + :c\,b:(w) + \cdots$$

and

$$\gamma(z)\beta(w) = \frac{1}{z-w} + :\gamma\,\beta:(w) + \cdots ;$$

and the energy-momentum tensors are

$$L_{bc} = 2\,:\partial c\,b: + :c\,\partial b: \tag{3.1}$$

$$L_{\beta\gamma} = -\frac{1}{2}\,:\gamma\,\partial\beta: - \frac{3}{2}\,:\partial\gamma\,\beta: . \tag{3.2}$$

The central charges of the corresponding Virasoro representations are $c_{bc} = -26$ and $c_{\beta\gamma} = 11$. In addition to the energy-momentum tensor $L_{\text{gh}} = L_{bc} + L_{\beta\gamma}$, which is the variation of the ghost action w.r.t. the graviton field, we need to introduce the supercurrent $G_{\text{gh}}$ given by the variation of the ghost action w.r.t. the gravitino

$$G_{\text{gh}} = -2\,:c\,\partial\beta: - 3\,:\partial c\,\beta: + :\gamma\,b:$$

Operators $L_{\text{gh}}$ and $G_{\text{gh}}$ together generate a super Virasoro algebra with the central charge $\hat{c} = (2/3)(c_{bc} + c_{\beta\gamma}) = -10$.

### 3.1.2 Bosonization of superconformal ghosts

Fermionic ghosts $b$ and $c$ can be bosonized using the following procedure. One has to introduce a bosonic field $\sigma$ such that

$$j_\sigma = \partial\sigma = -\,:b\,c: . \tag{3.3}$$

The energy momentum tensor $L_{bc}$ can be rewritten in term of $j_\sigma$ as

$$L_{bc} = \frac{1}{2} :j_\sigma^2: +3\,\partial j_\sigma, \tag{3.4}$$

which can be proven by a direct substitution of the r.h.s. of eq. (3.3) and comparison with eq. (3.1). Fermionic fields $b$ and $c$ can be recovered back from $\sigma$ as

$$c = e^\sigma \qquad\qquad b = e^{-\sigma}.$$

If we proceed similarly for the bosonic $\beta$ and $\gamma$ and, following the sign convention of [3], introduce

$$j_\phi = -\partial\phi = - :\beta\,\gamma:$$

the energy momentum tensor constructed out of $j_\phi$,

$$L_\phi = -\frac{1}{2} :j_\phi^2: +\partial j_\phi, \tag{3.5}$$

would not reproduce $L_{\beta\gamma}$ because of an extra term which comes out of $:j_\phi^2:$

$$:j_\phi^2:=:\beta\,\partial\gamma: - :\partial\beta\,\gamma: +\frac{1}{2}(:\beta^2\,\gamma^2: + :\gamma^2\,\beta^2:).$$

Note that if $\beta$ and $\gamma$ were fermions, this term would be zero as it is for $b\,c$ ghosts. Therefore instead of eq. (3.4) we now have

$$L_{\beta\gamma} = L_\phi - \frac{1}{4}(:\beta^2\,\gamma^2: + :\gamma^2\,\beta^2:).$$

Furthermore, the exponents $e^\phi$ and $e^{-\phi}$ cannot reproduce $\beta$ and $\gamma$ since they are fermions and their operator product expansion is

$$:e^{\phi(z)}::e^{-\phi(w)}:= (z-w)\ :e^{\phi(z)-\phi(w)}:= (z-w) + (z-w)^2\partial\phi + \cdots . \tag{3.6}$$

It turns out that $e^\phi$ and $e^{-\phi}$ correspond to the delta functions of the bosonic ghosts

$$\delta(\beta) = e^\phi \qquad\qquad \delta(\gamma) = e^{-\phi}.$$

In order to find the bosonization formulae for $\beta$ and $\gamma$ one has to introduce an auxiliary system of free fermions $\eta$ and $\xi$ with conformal weights 1 and 0 such that

$$\eta = \partial\gamma\,\delta(\gamma) \qquad\qquad \partial\xi = \partial\beta\,\delta(\beta).$$

The extra term in eq. (3.5) can be identified with the energy-momentum tensor of the $\eta\,\xi$ system:

$$L_{\eta\xi} = -\eta\partial\xi = -\frac{1}{4}(:\beta^2\,\gamma^2: + :\gamma^2\,\beta^2:).$$

The auxiliary fermions can be bosonized in the same manner as the $b\,c$ ghosts using

$$j_\chi = \partial\chi = -:\xi\,\eta: \qquad\qquad \xi = e^{\chi} \qquad\qquad \eta = e^{-\chi}.$$

The bosonic ghosts can now be recovered through the following formulae

$$\beta = e^{-\phi}\partial\xi = e^{-\phi+\chi}\partial\chi \qquad\qquad \gamma = e^{\phi}\eta = e^{\phi-\chi}$$

Bosonization formulae are extremely useful in calculations, but they tend to hide the geometrical meaning of the superconformal ghosts. Since exposing the geometry is our primary goal in this paper, we will use the bosonization formulae only to show the relation between the new geometric formalism and the original approach.

### 3.1.3 The BRST operator

When the superconformal ghosts are combined with a $\hat{c} = 10$ matter representation of the super Virasoro algebra $L_{\mathrm{m}}$, $G_{\mathrm{m}}$, one can define a nilpotent BRST operator as

$$Q = Q^{(0)} + Q^{(1)} + Q^{(2)} \tag{3.7}$$

where

$$\begin{aligned} Q^{(0)} &= \oint \frac{\mathrm{d}z}{2\pi i} :c(z)\,(L_{\mathrm{m}}(z) + L_{\beta\gamma}(z) + \frac{1}{2}L_{bc}(z)): \\ Q^{(1)} &= -\oint \frac{\mathrm{d}z}{2\pi i}\gamma(z)\,G_{\mathrm{m}}(z) \\ Q^{(2)} &= -\oint \frac{\mathrm{d}z}{2\pi i}\gamma^2(z)\,b(z) \end{aligned}$$

The first term $Q^{(0)}$ has the same structure as the BRST operator of the bosonic string in the background described by the conformal field theory with the energy momentum tensor given by $L = L_{\mathrm{m}} + L_{\beta\gamma}$.

### 3.1.4 Picture changing operation

Bosonization formulae for the $\beta\gamma$ system do not contain the zero mode of the auxiliary fermion $\xi$. Therefore a state like $\xi_0|\,\psi\,\rangle$ does not belong to the superstring BRST complex. This fact was used by FMS to introduce their picture changing operation. Let $|\,\psi\,\rangle$ be a BRST closed state, then the state

$$\Gamma|\,\psi\,\rangle = Q\,\xi_0|\,\psi\,\rangle,$$

will not contain $\xi_0$ and will be BRST closed simply because $Q^2 = 0$. Although the definition of $\Gamma|\,\psi\,\rangle$ suggests that it is BRST trivial, it is not since $\xi_0|\,\psi\,\rangle$

does not belong to the BRST complex. The picture changing operator was first introduced in [4] as

$$\Gamma(z) = [Q, \xi(z)].$$

One can think of $\Gamma(z)$ as a conformal field corresponding to the state $\Gamma|\,0\,\rangle \equiv \Gamma(0)|\,0\,\rangle$.

## 3.2 New geometrical approach

### 3.2.1 Superconformal ghosts as semi-infinite forms

In our recent work [14], we have shown that the ghost sector of the superstring can be interpreted as a space of semi-infinite singular forms on the algebra of superconformal vector fields on $S^{1|1}$. Let $l(z)$ and $g(z)$ be the even and odd vector fields concentrated[4] at the point $z$

$$\begin{aligned} l(z) &= \delta(z'-z)\frac{\partial}{\partial z'} = \sum_{n=-\infty}^{\infty} \frac{l_n}{z^{n+2}}, \\ g(z) &= \delta(z'-z)\frac{\partial}{\partial z'} = \sum_{r=-\infty}^{\infty} \frac{g_r}{z^{r+\frac{3}{2}}}. \end{aligned}$$

The conjugate covectors we will denote by $l^\vee(z)$ and $g^\vee(z)$ and define by

$$\langle l^\vee(z), l(w)\rangle = \delta(z-w) \qquad \langle g^\vee(z), g(w)\rangle = \delta(z-w).$$

Conformal fields of the superconformal ghosts can now be written as inner and exterior product operators.[5]

$$b(z) = \mathbf{i}_{l(z)} \qquad \beta(z) = \mathbf{i}_{g(z)} \tag{3.8}$$

$$c(z) = \mathbf{e}_{l^\vee(z)} \qquad \gamma(z) = \mathbf{e}_{g^\vee(z)} \tag{3.9}$$

The operator product expansions for $b$, $c$, $\beta$, $\gamma$, $\delta(\beta)$ and $\delta(\gamma)$ now follow easily from the properties of $\mathbf{i}$, $\mathbf{e}$ and delta functions of them. For example eqs. (2.10) and (2.10) translate into (*c.f.* eq. (3.6))

$$\delta(\beta(z))\delta(\gamma(w)) = (z-w) - (z-w)^2 :\beta\,\gamma:(w) + \cdots$$

and eqs. (2.9)—to

$$\delta(\beta(z))\delta(\beta(w)) = \frac{1}{z-w} :\delta(\partial\beta)\delta(\beta):(w) + \frac{1}{2} :\partial^2\beta\delta'(\partial\beta)\delta(\beta):(w) + \cdots$$

and

[4] The delta function in the following formulae are should be interpreted as distributions on the unit circle and properly normalized.

[5] Here we use a normalization for the bosonic ghosts which is slightly different form that of Friedan, Martinec and Shenker [3]. Our $\beta$ is $2\,\beta$ in the FMS notation and our $\gamma$ is $\frac{1}{2}\gamma$.

$$\delta(\gamma(z))\delta(\gamma(w)) = \frac{1}{z-w} :\delta(\partial\gamma)\delta(\gamma): (w) + \frac{1}{2} :\partial^2\gamma\delta'(\partial\gamma)\delta(\gamma): (w) + \cdots$$

The ghost vacua can be identified with the semi-infinite forms using the identity

$$\exp(r\,\sigma - s\,\phi)|\,0\,\rangle = |\,r|s\,\rangle, \tag{3.10}$$

where $|\,r|s\,\rangle$ is a semi-infinite form of degree $r|s$ given by

$$|\,r|s\,\rangle = \tilde{l}^{\vee}_{-r+2}\,\delta(\tilde{g}^{\vee}_{-s+\frac{3}{2}})\,\tilde{l}^{\vee}_{-r+3}\,\delta(\tilde{g}^{\vee}_{-s+\frac{5}{2}})\;\cdots\,.$$

Together with eqs. (3.8) and (3.9), this identity provides the translation between the language of FMS and the language of semi-infinite forms.

### 3.2.2 The BRST complex

In order for the differential which is also known as the BRST operator $Q$ to be nilpotent the semi-infinite forms should take values in a matter representation of the super Virasoro algebra with the central charge $\hat{c} = 10$. Let $L_{\mathrm{m}}(z)$ and $G_{\mathrm{m}}(z)$ be the super Virasoro generators in the matter representation, then we can write the Lie derivative operator on the semi-infinite forms as

$$\begin{aligned}\mathcal{L}_{l(z)} &= L(z) = L_{\mathrm{gh}}(z) + L_{\mathrm{m}}(z)\\ \mathcal{L}_{g(z)} &= G(z) = G_{\mathrm{gh}}(z) + G_{\mathrm{m}}(z)\end{aligned}$$

The BRST operator appears a differential for these semi-infinite forms and coincides with the one defined in eq. (3.7).

### 3.2.3 Picture changing operator(s)

Picture changing operation on the superstring BRST complex can be introduced in two ways: using the language of semi-infinite forms or the language of conformal fields. The two approaches are equivalent and they are related by the usual CFT correspondence of states and conformal fields. In the language of semi-infinite forms, one can associate a picture changing operator with every odd generator $g_r$ of the super Virasoro algebra using eq. (2.17)

$$\Gamma_{g_r} = \frac{1}{2}(G_r\,\delta(\beta_r) - \delta(\beta_r)\,G_r) = \delta(\beta_r)\,G_r + \delta'(\beta_r)\,b_{2r}$$

Alternatively one can introduce one conformal field of conformal dimension zero

$$\Gamma(z) = \Gamma_{g(z)} = \frac{1}{2}\big(\,:G\,\delta(\beta):(z) - \,:\delta(\beta)\,G:(z)\big) =:\delta(\beta)\,G:(z) + \,:\delta'(\beta)\,\partial b:(z).$$

Just like the picture changing operator of the singular de Rham complex, the conformal field $\Gamma$ can be formally written as a commutator with the differential

$$\Gamma(z) = [Q, \Theta(\beta(z))],$$

where $\Theta(\beta(z))$ is a fermionic field of conformal dimension zero. In the FMS formalism this field can be identified with the auxiliary fermion

$$\xi(z) = \Theta(\beta(z)).$$

This was first pointed out in [4].

The picture changing operator acts differently on different vacua. One can easily show that

$$\lim_{z \to 0} \Gamma(z) |\, r|s\, \rangle = \Gamma_{g_{-s+\frac{3}{2}}} |\, r|s\, \rangle.$$

This allows us to apply $\Gamma(z)$ repeatedly without having a picture changing operator with the same argument appear twice.

It was shown in [28, 29] that although the normal ordered product is neither commutative nor associative it generates a supercommutative and associative product in the cohomology called the dot product. Using the normal ordered product we can define the higher order picture changing operators as

$$\Gamma^{(n)} =: \Gamma^n : .$$

The on-shell associativity of the normal ordered product immediately shows that

$$\Gamma^{(n+1)} =: \Gamma\, \Gamma^{(n)} : + [Q, \cdots]. \tag{3.11}$$

This equation is the string version of eq. (2.20). Note that although the properties of the dot product make it trivial to prove eq. (3.11), they do not help in determining the BRST trivial part. This part can be easily deduced from eq. (2.19) or computed in a brute force calculation as in [5].[6]

### 3.2.4 Inverse picture changing operator

An inverse picture changing operator $Y$ was discovered in the bosonized form by the authors of [3] and further described in [7]. An explicit expression for $Y$ in terms of ghost fields was deduced in [22] to have the following simple form

$$Y(z) =: c(z)\, \delta'(\beta(z)) : . \tag{3.12}$$

This expression resembles the second term in eq. (2.22), but a closer look reveals that it is analogous to the whole expression for $Y_D$. To make the analogy complete we have to relate $\beta$ to $\mathbf{i}_D$ and $c$ to $\mathrm{d}x - \xi\, \mathrm{d}\xi$. Eq. (2.22) can be rewritten as

$$Y_D = (\mathrm{d}x - \xi\, \mathrm{d}\xi)\delta'(\mathrm{d}\xi),$$

[6] Note that in the work [5] a different definition of the normal ordering is implicitely used. The normal ordering of [5] is the symmetric normal ordering and the picture changing operator appears without the extra $\delta'(\beta)\partial b$ term. It is not clear though what prescription should be made there for the multiple normal ordered products.

which looks very similar to eq. (3.12). This analogy is not accidental as will be explained in section 4.

In the case of supermanifolds we were able to construct the inverse picture changing operator only for the $1|1$-dimensional case. Generalization to the higher dimensions would require a better understanding of the global structure of the supermanifold and may even not be possible. In the case of the superstring we can take advantage of the additional algebraic structure on the BRST cohomology supplied by the normal ordered product. As we mentioned above, the normal ordered product induces an associative dot product in the cohomology. Therefore in order to prove that $Y(z)$ is the inverse to the picture changing operator in the cohomology all we have to prove is that

$$[Y]\cdot[\Gamma] = 1 \tag{3.13}$$

where $[\phi]$ denotes the cohomology class of $\phi$. Indeed, it follows from the associativity and commutativity of the dot product and eq. (3.13) that

$$[Y]\cdot\big([\Gamma]\cdot[\phi]\big) = [\Gamma]\cdot\big([Y]\cdot[\phi]\big) = \big([Y]\cdot[\Gamma]\big)\cdot[\phi] = [\phi].$$

The identity (3.13) follows trivially from the fact that

$$:Y\,\Gamma:= 1. \tag{3.14}$$

Note that although eq. (3.14) is satisfied exactly without any BRST trivial term, it does not mean that taking a normal ordered product with $Y$ is an inverse operation to taking a normal ordered product with $\Gamma$. In order to prove this fact in cohomology we had to use associativity but the normal ordered product is not associative off shell.

### 3.2.5 Strongly physical states

Strongly physical states are closed states in BRST complex that have a form of a primary matter state multiplied by a standard ghost factor. In the Neveu-Schwarz sector these states can be represented by a conformal field

$$\varphi = c\,\delta(\gamma)\,V,$$

where $V(z)$ is a matter conformal field of dimension $1/2$, or equivalently by a state

$$|\,\varphi\,\rangle = c_1\delta(\gamma_{\frac{1}{2}})\,V_{-\frac{1}{2}}|\,0\,\rangle$$

Taking the normal ordered product $:\Gamma\,\varphi:$ is equivalent to evaluating $\Gamma_{g_{-\frac{1}{2}}}|\,\varphi\,\rangle$.

$$:\Gamma\,\varphi:= c\,[G_{-\frac{1}{2}}, V] + \gamma\,V \tag{3.15}$$

$$\Gamma_{g_{-\frac{1}{2}}}|\,\varphi\,\rangle = c_1\,G_{-\frac{1}{2}}V_{-\frac{1}{2}}|\,0\,\rangle + \gamma_{\frac{1}{2}}\,V_{-\frac{1}{2}}|\,0\,\rangle \tag{3.16}$$

One can easily see the analogy between these formulae and the $1|1$-dimensional result

$$\Gamma_D \,\mathrm{d}x\delta(\mathrm{d}\xi)\, f(x,\xi) = (\mathrm{d}x - \xi\,\mathrm{d}\xi)\, Df(x,\xi) + \mathrm{d}\xi\, f(x,\xi).$$

We already mentioned the analogy between $c$ and $\mathrm{d}x - \xi\,\mathrm{d}\xi$ in the previous section. Here we continue this analogy by identifying $\gamma$ with $\mathrm{d}\xi$ and $G_{-\frac{1}{2}}$ with $D$. Again, the reasons for this analogy to take place will be clarified in section 4.

The new matter vertex $[G_{-\frac{1}{2}}, V]$, which appears in the eq. (3.15) was originally presented as a result of the picture changing operation on the vertex $V$. This interpretation was particularly attractive since the state $[G_{-\frac{1}{2}}, V]$ has conformal dimension one, which calls for an analogy with the bosonic string. In the bosonic string strongly physical states have the form

$$\varphi_{\mathrm{bose}} = c\, V_{\mathrm{bose}},$$

where $V_{\mathrm{bose}}$ is a dimension one primary matter vertex. The corresponding state is annihilated by $Q^{(0)}$, but not $Q^{(1)}$ or $Q^{(2)}$ and therefore cannot be used as a superstring state. Another problem with this state is that it is annihilated by the inverse picture changing operator.

It is rather peculiar that the two terms in eq. (3.15) or (3.16) play very distinct roles in the calculations. The first term is the only term to contribute to amplitudes with other strong physical states but it vanishes under the inverse picture changing operation. The second term does not contribute to the amplitudes but it is the only one to reproduce the original state under the inverse picture changing operation.

# 4 From superstring to supermanifold

In this section we will tie together the results obtained so far by showing the relation between the superstring BRST complex and the de Rham complex of the super-moduli space. For each $n$-string state expressed as an element of an $n$-fold tensor product of the superstring BRST complexes we will define a corresponding singular form on the supermoduli space of decorated superconformal manifolds. The even and odd dimensions of the resulting differential forms will be determined by the gradings of the multi-string state. We will show that the action of the BRST operator on the multi-string state is equivalent to taking the de Rham differential of the corresponding form. Similar result will be obtained for the picture changing operators.

## 4.1 Superstring forms

As it was shown in [13], for any $n$-string state $|\,\Psi\,\rangle$ given as an element of the $n$-fold tensor power of the BRST complex one can define a differential form on the moduli space of decorated superconformal surfaces $\mathscr{P}_{g,n}$. Let $|\,\Psi\,\rangle$ have

the superghost number $-\deg(\langle\,\Sigma\,|) + r|s$, then the corresponding form will have degree $r|s$ and its value on $r$ even and $s$ odd vectors can be computed as

$$\Omega_\Psi(v_1, v_2, \ldots, v_r|\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_s) = \\ \langle\,\Sigma\,|B(v_1)\,B(v_2)\cdots B(v_r)\,\delta(B(\hat{v}_1))\,\delta(B(\hat{v}_2))\cdots\delta(B(\hat{v}_s))|\,\Psi\,\rangle,$$

where $\langle\,\Sigma\,|$ is the state defined by the superconformal field theory of ghosts and matter for the surface $\Sigma \in \mathscr{P}_{g,n}$. The superghost number of $\langle\,\Sigma\,|$ is determined by its genus and the number of Neveu-Schwarz and Ramond punctures

$$\deg(\langle\,\Sigma\,|) = (3\,g - 3 + n_{\rm NS} + n_{\rm R})|(2\,g - 2 + n_{\rm NS} + \frac{1}{2}n_{\rm R}).$$

Operators $B(v)$ are the standard antighost insertions. The antighost incertions can be evaluated by representing a tangent vector $v \in T_\Sigma\mathscr{P}_{g,n}$ by a Schiffer variation $v = (v^{(1)}, \cdots, v^{(n)})$, where $v^{(i)} = v_0^{(i)}(w_i) + \eta_i\, v_1^{(i)}(w_i)$ is a meromorphic superconformal vector field in the $i$-th coordinate patch

$$B(v) = \sum_{i=1}^{n} \Big(\oint \mathrm{d}w_i\; b^{(i)}(w_i)\, v_0^{(i)}(w_i) + \oint \mathrm{d}w_i\; \beta^{(i)}(w_i)\, v_1^{(i)}(w_i) Big).$$

Similarly we define $T(v)$ as

$$T(v) = [Q, B(v)] = \sum_{i=1}^{n} \Big(\oint \mathrm{d}w_i\; L^{(i)}(w_i)\, v_0^{(i)}(w_i) + \oint \mathrm{d}w_i\; G^{(i)}(w_i)\, v_1^{(i)}(w_i)\Big),$$

where $Q = \sum_{i=1}^n Q^{(i)}$ is the BRST operator on the $n$-th tensor power of the BRST complex.

It follows easily from the definition of the de Rham differential and the conformal field theory property

$$\delta_v\langle\,\Sigma\,| = \langle\,\Sigma\,|T(v),$$

that the BRST operator acting on $\Psi$ corresponds to taking the de Rham differential of $\Omega_\Psi$

$$\mathbf{d}\Omega_\Psi = \Omega_{Q\,\Psi}.$$

Similarly operator $B(v)$ corresponds to the inner product and $T(v)$ to the Lie derivative

$$\mathbf{i}_v\,\Omega_\Psi = \Omega_{B(v)\,\Psi}, \tag{4.1}$$

$$\delta(\mathbf{i}_{\hat{v}})\,\Omega_\Psi = \Omega_{\delta(B(\hat{v}))\,\Psi}, \tag{4.2}$$

$$\mathcal{L}_v\,\Omega_\Psi = \Omega_{T(v)\,\Psi}. \tag{4.3}$$

The picture changing operator defined in section 3.2.3, can be generalized for the $n$-th tensor power of the BRST complex to become

$$\Gamma_{\hat{v}} = \frac{1}{2}\big(\delta(B(\hat{v}))\,T(\hat{v}) - T(\hat{v})\,\delta(B(\hat{v}))\big) = \delta(B(\hat{v}))\,T(\hat{v}) + \delta(B(\hat{v}))\,(T(\hat{v}))^2. \tag{4.4}$$

It follows from eqs. (4.2) and (4.3) that

$$\Gamma_{\hat{v}} \Omega_\Psi = \Omega_{\Gamma_{\hat{v}} \Psi}, \tag{4.5}$$

where $\Gamma_{\hat{v}}$ in the right hand side should be evaluated according to eq. (4.4).

Finally a covector $v^\vee \in T^*_\Sigma \mathscr{P}_{g,n}$ can be represented by an $n$-tuple of meromorphic superconformal fields of dimension $3/2$, $v^{\vee(i)} = v_0^{\vee(i)}(w_i) + \eta_i\, v_1^{\vee(i)}(w_i)$, which can be used to define an operator

$$C(v^\vee) = \sum_{i=1}^{n} \Big( \oint \mathrm{d}w_i \; c^{(i)}(w_i)\, v_0^{\vee(i)}(w_i) + \oint \mathrm{d}w_i \; \gamma^{(i)}(w_i)\, v_1^{(i)}(w_i) \Big),$$

which corresponds to the exterior product operator

$$\mathbf{e}_{v^\vee} \Omega_\Psi = \Omega_{C(v^\vee)\,\Psi},$$
$$\delta(\mathbf{e}_{\hat{v}^\vee}) \Omega_\Psi = \Omega_{\delta(C(\hat{v}^\vee))\,\Psi}.$$

## 4.2 States to fields correspondence

States to fields correspondence is a familiar phenomenon for conformal field theories. Let us demonstrate how this correspondence works when the states are taken from the BRST complex of the superstring. In this case it becomes more natural to associate an operator valued differential form rather than an operator valued function with a state which has a non-zero ghost number.

Let $S_2 \in \mathscr{P}_{0,2}$ be a super sphere with two NS punctures and local coordinates given by $w_1 = z$, $\eta_1 = \theta$ and $w_2 = -1/z$, $\eta_2 = \theta/z$, where $(z, \theta)$ are the uniformizing coordinates and the punctures are located at $(0,0)$ and $(\infty, 0)$. The corresponding SCFT state defines a non-degenerate pairing on the BRST complex. Using this pairing we can turn a state associated with a 3-punctured sphere with one inserted state $\Phi$ into an operator $\Phi(S_3)$ with the matrix values

$$\langle\, A \,|\Phi(S_3)|\, B \,\rangle = \langle\, S_3 \,|\, A \otimes \Phi \otimes B \,\rangle,$$

where $\langle\, A \,|$ is defined by

$$\langle\, A \,|\, B \,\rangle = \langle\, S_2 \,|\, A \otimes B \,\rangle.$$

Equivalently we can define an operator valued form

$$\langle\, A \,|\hat{\Omega}_\Phi|\, B \,\rangle = \Omega_{A \otimes \Phi \otimes B}.$$

Let us define the map from the super complex plain $\mathbb{C}^{1|1}$ to $\mathscr{P}_{0,3}$ by mapping a point on $\mathbb{C}^{1|1}$ with coordinates $(z, \theta)$ to a sphere with punctures at $(0,0)$, $(z, \theta)$ and $(\infty, 0)$ and the local coordinates given by

$$(w_1, \eta_1) = (z', \theta')$$
$$(w_2, \eta_2) = (z' - z - \theta'\,\theta, \theta' - \theta)$$
$$(w_3, \eta_3) = (-1/z', \theta'/z')$$

where $z', \theta'$ are the uniformizing coordinates. Using this map we can pull back the form $\hat{\Omega}_\Phi$ from $\mathscr{P}_{0,3}$ to $\mathbb{C}^{1|1}$ and define the form $\Phi(\boldsymbol{z})$ associated with the state $|\,\Phi\,\rangle$.

Let us now do some examples.

$$
\begin{aligned}
&|\,\Phi\,\rangle = V(0)|\,0\,\rangle && \Phi(\boldsymbol{z}) = V(\boldsymbol{z}) \\
&|\,\Phi\,\rangle = c_1 V(0)|\,0\,\rangle && \Phi(\boldsymbol{z}) = (\mathrm{d}z - \theta\,\mathrm{d}\theta)\,V(\boldsymbol{z}) \\
&|\,\Phi\,\rangle = c_1 \delta(\gamma_{\frac{1}{2}}) V(0)|\,0\,\rangle && \Phi(\boldsymbol{z}) = \mathrm{d}z\,\delta(\mathrm{d}\theta)\,V(\boldsymbol{z}),
\end{aligned}
$$

where $V(z)$ is the matter only vertex and $V(\boldsymbol{z})$ is defined as

$$
V(\boldsymbol{z}) = V(z,\theta) = V(z) + \theta\,[G_{-\frac{1}{2}}, V(z)].
$$

This is exactly the same superconformal vertex operator as the one used in FMS. In particular $V(\boldsymbol{z})$ satisfies

$$
[G_{-\frac{1}{2}}, V(\boldsymbol{z})] = DV(\boldsymbol{z})
$$

This construction explains the analogy between the BRST complex and the de Rham complex of $S^{1|1}$ which we observed several times in the preceding discussion.

# 5 Conclusion and outlook

We have shown that the picture changing operation is not just a bizzare feature of the superstring theory but is natural ingredient of the geometrical integration theory on the supermanifolds. This interpretation allows one to rederive many properties of the picture changing operators which were observed over the years in a consistent manner. The geometric description naturally explains the origin of singularities that appear in the product of two picture changing operators and allows to define the higher order picture changing operators. The higher order picture changing operators still present some puzzles to be solved. We introduced order $n$ operators in section 2.6 simply as symmetrized products of first order picture changing operators associated with $n$ linearly independent odd vector fields $\hat{v}_i$

$$
\Gamma^{(n)}_{\hat{v}_1\cdots\hat{v}_n} = \sum_{\sigma\in\mathfrak{S}_n} \Gamma_{\hat{v}_{\sigma(1)}} \cdots \Gamma_{\hat{v}_{\sigma(n)}}.
$$

There is a strong evidence that $\Gamma^{(n)}_{\hat{v}_1\cdots\hat{v}_n}$ does not change if the set of odd vector fields $\{\hat{v}_1 \cdots \hat{v}_n\}$ is replaced by another set which spans the same $n$-dimensional space at every point of the supermanifold. It is relatively simple to verify this hypothesis for $n = 1$ and $n = 2$ and for constant vector fields with a straightforward calculation. This suggests that the higher order PCOs should be interpreted as an integration over the subspace spanned by $\{\hat{v}_1 \cdots \hat{v}_n\}$. This is easy to check if the vector fields have vanishing Lie brackets with each other and with

themselves, but in general it is hard to define such “partial integration”. The main problem is that by exponentiating odd vector fields with non-vanishing Lie brackets one introduces a flow in even directions as well as odd ones.

There still remain some puzzles with the off-shell generalization of the picture changing operators. It is not clear yet what is the significance of the BRST exact terms in the product of two PCOs. It is likely that they may play a prominent role in the superstring field theory.

We did not analyze the relation between the picture changing operation and the space-time supersymmetry. In [14] we demonstrated that the SUSY algebra can be constructed geometrically without picture changing, but the question whether the resulting algebra commutes with the PCOs off-shell is still unclear.

Our analysis was done entirely within the holomorphic sector of the superstring, but it should be straightforward to generalize it to the full type II superstring. Further understanding of the off-shell structure of the type II superstring BRST complex will hopefully shed some light to the long standing problem of constructing the full string field action for the Ramond-Ramond sector. Some progress in this direction has been achieved recently in [30–32].

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